

# Graph pegging numbers

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## Abstract

In graph pegging, we view each vertex of a graph as a hole into which a peg can be placed, with checker-like “pegging moves” allowed. Motivated by well-studied questions in graph pebbling, we introduce two pegging quantities. The pegging number (respectively, the optimal pegging number) of a graph is the minimum number of pegs such that for every (respectively, some) distribution of that many pegs on the graph, any vertex can be reached by a sequence of pegging moves. We prove several basic properties of pegging and analyze the pegging number and optimal pegging number of several classes of graphs, including paths, cycles, products with complete graphs, hypercubes, and graphs of small diameter.

## 1 Introduction

The mathematics of peg jumping originated in the games of Peg Solitaire and Conway’s Soldiers (see Berlekamp, Conway, and Guy [1]). Much has been written about answering the classical questions of Peg Solitaire and Conway’s Soldiers in more general settings (see, for example, Eriksen, Eriksson, and Eriksson [5]). In this paper, however, we consider peg jumping – or, as we call it, pegging – on graphs, and our questions are inspired by work in the theory of graph pebbling.

Given a graph, we view each vertex as a hole into which one peg can be placed. A *pegging move* consists of removing two pegs from adjacent holes and placing one peg in a third, empty hole adjacent to one of the first two holes. In essence, one peg is jumping the other and landing in the third hole (with the jumped peg being removed). If there are pegs in some of the vertices of the graph, we say that we can *peg to a vertex* if we can move a peg to that vertex with a (possibly empty) sequence of pegging moves.

Conway's Soldiers can be recast in this general setting. In that game, the graph in question is embedded in the Cartesian plane with vertex set  $\mathbb{Z}^2$ , and there is an edge between two vertices if their Euclidean distance is 1. Pegs are placed at all vertices in the lower half-plane, and the challenge is to move a peg as far as possible into the upper half-plane by peg-jumping. Classically, however, a peg is only allowed to jump in a straight line (horizontally or vertically) over another peg, which is more restrictive than our pegging moves. In fact, whether or not all pegging moves are allowed, the best possible solution moves a peg up four units. This was proved in the classical case by Conway using a weight argument; our results on graphs use an extension of that weight argument and incidentally show that the optimum solution to Conway's Soldiers cannot be improved upon using pegging moves.

As mentioned, the pegging questions we consider are motivated by graph pebbling (see Hurlbert [7] for a survey of graph pebbling). The *pegging number* of a graph is the minimum number of pegs so that no matter how those pegs are distributed on the graph, we can peg to any vertex. In contrast, the *optimal pegging number* of the graph is the minimum number of pegs so that there is *some* way to distribute those pegs on the graph so that we can peg to any vertex. These definitions mirror those of the pebbling number and the optimal pebbling number of a graph. In fact, we will use results on pebbling numbers to prove results on pegging numbers; basic pebbling definitions will be given when needed.

A formal definition of pegging and fundamental pegging lemmas appear in Section 2. In Section 3, we study the pegging numbers of several classes of graphs, including paths and cycles. In Section 4, we move on to the Cartesian product of an arbitrary graph  $G$  with a complete graph  $K_n$ . In particular, Theorem 4.3 relates the pegging number of  $G \times K_n$  to the pebbling number of  $G$ . In Section 5, we apply this result to compute the pegging number of the hypercube, and we also obtain upper and lower bounds for the optimal pegging number of the hypercube; this uses the theory of binary linear codes. Finally, in Section 6, we give an upper bound for the optimal pegging number of a graph of diameter 2, classify graphs with pegging number at most 3, and give an upper bound for the pegging number of a graph of diameter at most 3.

The concept of pegging on a graph was introduced by the third author at the 1994 University of Minnesota Duluth Research Experience for Undergraduates (REU). He and several of the students at the program spent a weekend exploring basic properties of pegging. Their results are included here, along with our later work. Further work on pegging has been done by Wood [11] and Levavi [8]. It should also be mentioned that Niculescu and Niculescu [10] independently proposed the idea of pegging on graphs; however, their only result in that direction is the first conclusion of Lemma 2.1.

## 2 The Basics of Pegging

A *distribution*  $D$  of pegs on a graph  $G$  is any subset of  $V(G)$ . Through Lemma 2.1,  $G$  may be any graph; after Lemma 2.1, we only consider finite, simple graphs. If  $u$  and  $v$  are distinct, adjacent vertices in  $D$ , and  $w$  is a vertex adjacent to  $v$  that is not in  $D$ , then the *pegging move*  $m = u \xrightarrow{v} w$  replaces the distribution  $D$  with the distribution  $m(D) = D \setminus \{u, v\} \cup \{w\}$ . Sometimes, to emphasize that the conditions on  $u$ ,  $v$ , and  $w$  are met, we call  $m$  a *valid* pegging move (on  $D$ ). The set  $\{u, v\}$  is the *source* of  $m$ , and the vertex  $w$  is the *destination* of  $m$ . If  $M$  is a sequence of pegging moves starting at  $D$ , then we write  $M(D)$  for the final distribution. A vertex  $t$  is *reachable* from a distribution  $D$  if there is a finite sequence of pegging moves  $M$  with  $t \in M(D)$ . The *reach* of a distribution  $D$ , denoted  $\text{Reach}(D)$ , is the set of all vertices reachable from  $D$ .

The first of the four tools given in this section for analyzing pegging is a weight argument adapted from the solution to Conway's Soldiers given in [1]. It is used to show that a given vertex is not in the reach of a distribution. Given a distribution  $D$  on a graph  $G$  and a vertex  $t$ , define the *weight* of  $D$  with respect to  $t$  to be

$$\text{wt}_t(D) = \sum_{u \in D} \sigma^{d(u,t)},$$

where  $\sigma = (\sqrt{5} - 1)/2$  is the positive root of  $x^2 + x = 1$ , and  $d(u, t)$  is the distance in  $G$  from  $u$  to  $t$ . Note that if  $D$  or  $G$  is infinite,  $\text{wt}_t(D)$  may be infinite.

**Lemma 2.1** (Monotonicity of Weight). *Let  $D$  be a distribution on a graph  $G$ , and let  $D'$  be a distribution obtained from  $D$  by a finite sequence of pegging moves. Then*

$$\text{wt}_t(D') \leq \text{wt}_t(D)$$

for all  $t \in G$ . If  $\text{wt}_t(D) < 1$ , then  $t \notin \text{Reach}(D)$ .

*Proof.* Without loss of generality, we may assume that  $D' = m(D)$ , where  $m = u \xrightarrow{v} w$ . For any vertex  $t$ , we know  $d(u, t) \leq d(w, t) + 2$  and  $d(v, t) \leq d(w, t) + 1$ . Thus

$$\begin{aligned} \text{wt}_t(D') &= \text{wt}_t(D) - \sigma^{d(u,t)} - \sigma^{d(v,t)} + \sigma^{d(w,t)} \\ &\leq \text{wt}_t(D) - \sigma^{d(w,t)+2} - \sigma^{d(w,t)+1} + \sigma^{d(w,t)} \\ &= \text{wt}_t(D) - (\sigma^2 + \sigma - 1)\sigma^{d(w,t)} \\ &= \text{wt}_t(D). \end{aligned}$$

The second claim follows from the observation if  $t \in \text{Reach}(D)$ , then  $t$  is in some distribution  $D'$  obtained from  $D$  by a finite sequence of pegging moves. But then  $1 \leq \text{wt}_t(D') \leq \text{wt}_t(D)$ .  $\square$

Combining Lemma 2.1 with a short computation shows that the optimum solution to Conway's Soldiers is still four units when all pegging moves are allowed. For the rest of the paper, we will only consider finite, simple graphs  $G$ . Our principal goal is to study the following two pegging invariants.

**Definition.** The *pegging number* of a graph  $G$  is the smallest positive integer  $d$  such that *every* distribution of size  $d$  on  $G$  has reach  $V(G)$ . The *optimal pegging number*  $p(G)$  of  $G$  is the smallest positive integer  $d$  such that *some* distribution of size  $d$  of  $G$  has reach  $V(G)$ .

We can make some general observations about  $P(G)$  and  $p(G)$ . Obviously  $p(G) \leq P(G)$ . Let  $|G|$  denote the *order* of  $G$ , that is, the number of vertices of  $G$ . If  $G$  is disconnected, then

$$P(G) = |G| - \min_C (|C| - P(C))$$

and

$$p(G) = \sum_C p(C),$$

where the minimum and sum are taken over all connected components  $C$  of  $G$ . Thus, we will focus primarily on connected graphs.

Let  $\alpha(G)$  denote the *independence number* of a graph  $G$ , that is, the maximum cardinality of a set of pairwise non-adjacent vertices of  $G$ . Clearly  $\alpha(G) \leq P(G) \leq |G|$ . Furthermore, if  $G$  has at least one edge, then  $\alpha(G) + 1 \leq P(G)$ . (In the pebbling literature, graphs achieving equality in the corresponding lower bound for the pebbling number are said to be “of class 0” or “demonic”, so one might say that a graph  $G$  with  $P(G) = \alpha(G) + 1$  is “of pegging class 0” or “devilish”.) If no connected component of  $G$  is isomorphic to a star graph  $K_{1,k}$ , then  $P(G) \leq |G| - 1$ . If  $G$  has at least two vertices, then  $p(G) \geq 2$ , with equality if and only if there are two adjacent vertices that dominate  $G$ . Finally, a simple application of Lemma 2.1 proves the following proposition, which was discovered by various mathematicians at the 1994 University of Minnesota Duluth REU.

**Proposition 2.2.** *If a graph  $G$  has diameter  $d$ , then  $P(G) \geq d$ .*

*Proof.* Choose vertices  $u_0$  and  $u_d$  with  $d(u_0, u_d) = d$ , and let  $u_0, u_1, \dots, u_d$  be a path of length  $d$  from  $u_0$  to  $u_d$ . Let  $D = \{u_2, \dots, u_d\}$ . Then

$$\text{wt}_{u_0}(D) = \sigma^2 + \sigma^3 + \dots + \sigma^d < 1.$$

Thus, by Lemma 2.1,  $v_0 \notin \text{Reach}(D)$  and  $P(G) \geq d$ . □

Our goal is to be able to compute the pegging numbers and optimal pegging numbers of a variety of graphs. Our first computational tool is the weight argument given in Lemma 2.1. Our remaining three tools show that allowing the removal of pegs, “stacking moves”, and “pebbling moves” does not increase the reach of a distribution. This helps us get upper bounds on pegging and optimal pegging numbers because we can use all of these moves to show that the reach of a distribution really is all of  $V(G)$ . The definition of these moves and the proof of the main result require a more sophisticated view of pegging; in particular, we have to number the pegs so that they are distinguishable and do not get “mixed up” when stacked on a vertex.

Given a graph  $G$ , let  $S_G = \{u_i : u \in V(G), i \in \mathbb{Z}\}$ . We interpret  $u_i$  to indicate that peg  $i$  is on vertex  $u$ . Define a *multi-distribution*  $D$  of pegs on  $G$  to be a finite subset of  $S_G$

with the property that if  $u_i, v_i \in D$ , then  $u = v$  (that is, peg  $i$  can only be on one vertex at a time).

If  $u, v$ , and  $w$  are distinct vertices,  $v$  is adjacent to  $u$  and  $w$ , and  $u_i$  and  $v_j$  are in  $D$ , then the *stacking move*  $m = u_i \xrightarrow{v_j} w_i$  sends the multi-distribution  $D$  to the multi-distribution  $m(D) = D \setminus \{u_i, v_j\} \cup \{w_i\}$ . If  $u$  and  $w$  are distinct, adjacent vertices,  $i \neq j$ , and  $u_i$  and  $u_j$  are in  $D$ , then the *pebbling move*  $m = u_i \xrightarrow{u_j} w_i$  sends the multi-distribution  $D$  to the multi-distribution  $m(D) = D \setminus \{u_i, u_j\} \cup \{w_i\}$ . (See Section 4 for a discussion of pebbling.) Finally, if  $u_i$  is in  $D$ , then the *removal move*  $m$  (with respect to  $u_i$ ) sends the multi-distribution  $D$  to the multi-distribution  $m(D) = D \setminus \{u_i\}$ .

We view distributions (in which the pegs happen to be labeled) as multi-distributions in the obvious way, and we view pegging moves as stacking moves in the obvious way. To emphasize that a multi-distribution is, in fact, a distribution, we may refer to it as a *proper* distribution. If  $D$  is a proper distribution, let  $\text{Reach}_a(D)$  denote the set of vertices reachable from  $D$  via all moves (stacking, pebbling, and removal). Note that  $\text{Reach}_a(D) \supseteq \text{Reach}(D)$ .

Given a multi-distribution  $D$ , a *move forest* of  $D$  is a labeled binary forest (that is, a disjoint union of labeled binary trees) with the following three properties:

1. The label on each node is an element of  $S_G$ ; multiple nodes may have the same label.
2. The label on each leaf node is an element of  $D$ ; no two leaf nodes may have the same label.
3. Each interior node has a left child and a right child. If an interior node is labeled  $w_i$ , then either:
  - the left and right children have labels of the form  $u_i$  and  $v_j$ , respectively, where  $u, v$ , and  $w$  are distinct vertices,  $v$  is adjacent to  $u$  and  $w$ , and  $i \neq j$ , or
  - the left and right children have labels of the form  $u_i$  and  $u_j$ , respectively, where  $u$  and  $w$  are distinct, adjacent vertices and  $i \neq j$ .

A *traversal* of a move forest is an ordering of the interior nodes so that each interior node precedes its ancestors. Each traversal  $N = (n_1, n_2, \dots)$  of a move forest corresponds to a sequence of valid stacking and pebbling moves  $M = (m_1, m_2, \dots)$  on  $D$ , where  $m_r = u_i \xrightarrow{v_j} w_i$  if node  $n_r$ , its left child, and its right child are labeled  $w_i, u_i$ , and  $v_j$  respectively, and  $m_r = u_i \xrightarrow{u_j} w_i$  if node  $n_r$ , its left child, and its right child are labeled  $w_i, u_i$ , and  $u_j$  respectively. The following proposition is clear.

**Proposition 2.3.** *Given a multi-distribution  $D$ , the above correspondence gives a bijection between sequences of valid stacking and pebbling moves on  $D$  and traversals of move forests on  $D$ .*

**Theorem 2.4.** *Let  $D$  be a proper distribution on a graph  $G$ . Then  $\text{Reach}_a(D) = \text{Reach}(D)$ .*

*Proof.* The proof is by induction on the size of  $D$ . The result is clear when  $|D| = 1$ , so assume it is true for all distributions of size less than  $d$ , and let  $D$  be a distribution of size

d. Let  $t \in \text{Reach}_a(D) \setminus D$  and let  $M$  be a sequence of stacking, pebbling, and removal moves that puts a peg on  $t$ . If we were to remove all the removal moves from  $M$ , we would get a new sequence of stacking and pebbling moves on  $D$  that is valid and puts a peg on  $t$  (the only possible change is that some of the pegging moves in  $M$  change to non-pegging stacking moves). So we may assume  $M$  contains only stacking and pebbling moves. Let  $F$  be the move forest corresponding to  $M$ ; note that  $M$  has an interior node labeled  $t_a$  for some  $a \in \mathbb{Z}$ . By the inductive hypothesis, it suffices to show that there exists a valid pegging move  $m'$  on  $D$  so that  $t \in \text{Reach}_a(m'(D))$ .

Next we show that we can assume  $M$  has no pebbling moves. Suppose  $m = u_i \xrightarrow{u_j} w_i$  is a pebbling move in  $M$ , and let  $n$  be the node in  $F$  corresponding to  $m$ . Since  $D$  is proper, the left child of  $n$  (labeled  $u_i$ ) or the right child of  $n$  (labeled  $u_j$ ) is not a leaf node. We may assume without loss of generality that the left child  $n'$  of  $n$  is not a leaf node. (If the right child of  $n$  is not a leaf node, we can swap the left and right subtrees of  $n$  and change every occurrence of  $i$  in the labels of  $n$  and its ancestors to a  $j$  to form a new move forest that puts a peg on  $t$  in which the left child of  $n$  is not a leaf node.) If the right child of  $n'$  is labeled  $v_k$  for some vertex  $v \neq w$ , replace the left subtree of  $n$  by the right subtree of  $n'$  and change every occurrence of  $i$  in the labels of  $n$  and its ancestors to a  $k$ . If the right child of  $n'$  is labeled  $w_k$ , replace  $n$  and its subtree by the right subtree of  $n'$  and change every occurrence of  $i$  in the labels of the (former) ancestors of  $n$  to a  $k$ . In either case, we have a new move forest that puts a peg on  $t$  and has fewer pebbling moves than  $M$ . We can repeat this operation until all pebbling moves have been removed. Therefore, we may assume that  $M$  has no pebbling moves.

Finally we handle stacking moves. Let  $m$  be the first move in  $M$  whose target vertex is not in  $D$ . Write  $m = u_i \xrightarrow{v_j} w_i$ . Note that by our choice of  $m$ , the distribution  $D$  does contain  $u_k$  and  $v_l$  for some  $k$  and  $l$ , and  $m' = u_k \xrightarrow{v_l} w_k$  is a valid pegging move on  $D$ . We only need to show that  $t \in \text{Reach}_a(m'(D))$ . In fact, we will construct a new move forest  $F'$  with a node labeled  $t_b$  for some  $b \in \mathbb{Z}$  and an interior node labeled  $w_k$  whose left and right children, respectively, are leaf nodes labeled  $u_k$  and  $v_l$ . Then any traversal of  $F'$  starting with this interior node puts peg  $b$  at vertex  $t$  and shows  $t \in \text{Reach}_a(m'(D))$ .

Let  $n$  be the node in  $F$  corresponding to  $m$ . Let  $A$  be the left subtree of  $n$  and let  $B$  be the right subtree of  $n$ . Let  $C$  be the set of leaf nodes in  $F$  that are not leaves of  $A$  or  $B$ . Detach  $A$  and  $B$  from  $n$ , replacing  $A$  by a single leaf node labeled  $u_k$  and replacing  $B$  by a single leaf node labeled  $v_l$ .

There are some situations in which we will reattach  $A$  or  $B$  to a different part of the tree, and we describe those situations in this paragraph. If  $u_k$  is not the label of a leaf node in  $B$ , or  $v_j$  is not the label of a leaf node in  $A$ , we may assume without loss of generality that  $u_k$  is not the label of a leaf node in  $B$ . If  $u_k$  is the label of a leaf node in  $C$ , replace that leaf node by  $A$ . If  $v_l$  is the label of a leaf node in  $C$  or  $A$ , replace that leaf node by  $B$ .

Finally, choose any traversal of the current forest. Visiting each interior node in order, change the subscript on the label of the node to match the subscript on the label of its left child (if they are different). Let the resulting forest be  $F'$ ; we claim it is a move forest on  $D$ .

Property 1 in the definition of a move forest is clearly satisfied. Property 2 must be



satisfied since we only added leaf nodes labeled  $u_k$  and  $v_l$  and removed any other leaf nodes with those labels. Property 3 was also enforced in the previous paragraph. Furthermore,  $F'$  has the interior node  $n$  labeled  $w_k$  whose left and right children, respectively, are leaf nodes labeled  $u_k$  and  $v_l$ . Finally, the node labeled  $t_a$  in  $F$  must exist in  $F'$ , but possibly with the label changed to  $t_b$ . As mentioned, any traversal of  $F'$  starting with  $n$  shows  $t \in \text{Reach}_a(m'(D))$ , proving the theorem.  $\square$

**Corollary 2.5** (Monotonicity of Reach). *Let  $D' \subset D$  be two distributions on a graph  $G$ . Then  $\text{Reach}(D') \subseteq \text{Reach}(D)$ . If  $D$  is a distribution of size  $d$  and  $\text{Reach}(D) \neq V(G)$ , then  $P(G) > d$ . If  $\text{Reach}(D) \neq V(G)$  for every distribution  $D$  of size  $d$ , then  $p(G) > d$ .*

### 3 Paths, Cycles, and Joins

In this section, our goal is to compute the pegging and optimal pegging numbers of several simple classes of graphs. We will use  $K_n$  to denote the complete graph on  $n$  vertices;  $P_n$  to denote the path on  $n$  vertices; and  $C_n$  to denote the cycle on  $n$  vertices. For convenience, we will label the vertices of both  $P_n$  and  $C_n$  by  $v_1, v_2, \dots, v_n$  with  $v_i$  adjacent to  $v_{i+1}$  for  $1 \leq i \leq n-1$ . The complete graphs are simple to analyze:  $P(K_n) = p(K_n) = 2$  for  $n \geq 2$ . As for cycles,  $P(C_3) = 2$  and  $P(C_4) = 3$ , so we turn to cycles on five or more vertices.

**Theorem 3.1.** *For  $n \geq 5$ , the pegging number of the cycle  $C_n$  is  $P(C_n) = n - 2$ .*

*Proof.* In every distribution of  $n - 2$  pegs on  $C_n$ , for each of the two holes, there are two adjacent pegs with one of them adjacent to the hole, so that each hole is the destination for some move. Hence  $P(C_n) \leq n - 2$ .

On the other hand, let  $D = \{v_2, v_3, \dots, v_{n-2}\}$ . Suppose  $v_n \in \text{Reach}(D)$ , and let  $M$  be a minimum-length sequence of pegging moves with  $v_n \in M(D)$ . By minimality, the last move of  $M$  is the first move placing a peg at  $v_n$ , and by symmetry, we may assume that the last move of  $M$  is  $v_{n-2} \xrightarrow{v_{n-1}} v_n$ . Then  $M$  is a valid pegging sequence on the distribution  $D' = \{v_2, v_3, \dots, v_{n-2}\}$  on the path  $P_n$ . This says  $v_n \in \text{Reach}(D')$  while  $\text{wt}_{v_n}(D') < 1$ , contradicting Lemma 2.1. So  $\text{Reach}(D) \neq V(C_n)$  and  $P(C_n) > n - 3$ .  $\square$

**Theorem 3.2.** *For  $n \geq 3$ , the optimal pegging number of the cycle  $C_n$  is  $p(C_n) = \lceil n/2 \rceil$ .*

*Proof.* Let  $D$  be the distribution  $\{v_2, v_3, v_5, v_7, v_9, \dots, v_n\}$  if  $n$  is odd and  $\{v_2, v_3, v_5, v_7, v_9, \dots, v_{n-1}\}$  if  $n$  is even. Then  $R(D) = V(C_n)$  and  $|D| = \lceil n/2 \rceil$ , so  $p(C_n) \leq \lceil n/2 \rceil$ .

Suppose there is a distribution  $D$  of  $\lceil n/2 \rceil - 1$  pegs with  $\text{Reach}(D) = V(C_n)$ .  $D$  consists of blocks of consecutive pegs alternating with blocks of consecutive empty vertices. If  $D$  has a block of three or more empty vertices, then (by symmetry) we may assume that  $D$  is contained in the distribution  $\{v_2, v_3, \dots, v_{n-2}\}$ , whose reach is not  $V(C_n)$  by the proof of Theorem 3.1. By Corollary 2.5, this contradicts our choice of  $D$ , so  $D$  does not contain a block with three or more empty vertices.

$D$  has strictly more empty vertices than pegs. Empty vertices appear in blocks of one or two, so there are more blocks of two empty vertices than blocks of two or more pegs. In

particular, since  $D$  must have at least one block of two or more pegs, there are at least two blocks of two empty vertices. It follows that there exist empty vertices  $u$ ,  $v$ ,  $w$ , and  $x$  so that  $u$  and  $v$  are adjacent,  $w$  and  $x$  are adjacent, and the pegs in one of the two components  $G_1$  and  $G_2$  of  $C_n \setminus \{(u, v), (w, x)\}$  only come in blocks of one. Without loss of generality, let it be  $G_1$ , with  $v$  and  $w$  adjacent to  $G_1$ . No moves among the pegs in  $G_1$  are possible, since none are adjacent, until a sequence of moves on the pegs in  $D' = D \cap G_2$  puts a peg on  $v$  or  $w$ . However,  $\text{wt}_v(D') < 1$  and  $\text{wt}_w(D') < 1$ , so  $v, w \notin \text{Reach}(D')$ . Thus  $v$  and  $w$  are not in  $\text{Reach}(D)$ , contradicting our choice of  $D$ . The result follows.  $\square$

Using the optimal pegging number of the cycle  $C_n$  and the following proposition, we can compute  $P(P_n)$  and  $p(P_n)$ .

**Proposition 3.3.** *If  $H$  is a spanning subgraph of  $G$ , then  $p(G) \leq p(H)$  and  $P(G) \leq P(H)$ .*

*Proof.* Any move made on a distribution on  $H$  can be made on the same distribution on  $G$ , so the reach of a distribution on  $G$  contains the reach of the same distribution on  $H$ . The desired inequalities follow.  $\square$

For  $1 \leq n \leq 3$ , we clearly have  $P(P_n) = n$ , so we now consider the pegging numbers of paths of order at least 4.

**Theorem 3.4.** *For  $n \geq 4$ , the pegging number of the path  $P_n$  is  $P(P_n) = n - 1$ .*

*Proof.* It is clear that  $\text{Reach}(D) = V(P_n)$  for any distribution  $D$  of  $n - 1$  pegs on  $P_n$ . On the other hand, the diameter of  $P_n$  is  $n - 1$ , so by Proposition 2.2, we have  $P(P_n) \geq n - 1$ . Thus  $P(P_n) = n - 1$ .  $\square$

Obviously  $p(P_n) = n$  for  $n = 1$  and  $2$ , so we now consider larger values of  $n$ .

**Theorem 3.5.** *For  $n \geq 3$ , the optimal pegging number of the path  $P_n$  is  $p(P_n) = \lceil n/2 \rceil$ .*

*Proof.* Since  $P_n$  is a spanning subgraph of  $C_n$ , Proposition 3.3 shows that  $p(P_n) \geq p(C_n) = \lceil n/2 \rceil$ . On the other hand, let  $D$  be the distribution  $\{v_2, v_3, v_5, v_7, \dots, v_n\}$  if  $n$  is odd and  $\{v_2, v_3, v_5, v_7, \dots, v_{n-1}\}$  if  $n$  is even. Then  $\text{Reach}(D) = P_n$ , so  $p(P_n) \leq \lceil n/2 \rceil$ . Hence equality holds.  $\square$

We close this section by calculating the pegging and optimal pegging numbers of joins. The *join* of two graphs  $G$  and  $H$ , denoted  $G + H$ , has vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H) \cup \{(g, h) : g \in G, h \in H\}$ . Recall that  $\alpha(G)$  is the independence number of a graph  $G$ .

**Theorem 3.6.** *Given any two graphs  $G$  and  $H$  each having at least one vertex, the join  $G + H$  satisfies  $p(G + H) = 2$  and*

$$P(G + H) = \alpha(G + H) + 1 = \max(\alpha(G), \alpha(H)) + 1.$$



*Proof.* Clearly 2 is a lower bound for  $p(G + H)$ . To achieve this lower bound, place one peg on a vertex of  $G$  and one peg on a vertex of  $H$ . It is easy to see that the reach of this distribution is all of  $G + H$ , giving  $p(G + H) = 2$ .

Take  $a = \max(\alpha(G), \alpha(H))$ . Note that  $\alpha(G + H) = a$ , so  $P(G + H) \geq a + 1$ . To show that  $P(G + H)$  is exactly  $a + 1$ , first observe that if a distribution has one peg on a vertex of  $G$  and one peg on a vertex of  $H$ , then the reach is the entire graph. Let  $D$  be a distribution of  $a + 1$  pegs on  $G + H$  in which either all the pegs are on  $G$  or all the pegs are on  $H$ . Without loss of generality, let all the pegs be on  $G$ .

If  $a = 1$ , then  $G$  and  $H$  are both complete graphs and so  $G + H$  is complete. Since  $|D| = 2$ , it follows that  $\text{Reach}(D) = G + H$ . If  $a > 1$ , then  $|D| \geq 3$ . Since  $|D| > \alpha(G)$ , there are adjacent vertices  $u$  and  $v$  in  $D$ . Choose any vertex  $w$  in  $H$  and let  $m = u \xrightarrow{v} w$ . Then in the distribution  $m(D)$  there is a peg on a vertex of  $H$  and there remains a peg on a vertex of  $G$ , so every vertex is in the reach of  $m(D)$ . Hence we have  $\text{Reach}(D) = G + H$  for any distribution of  $a + 1$  pegs on  $G + H$ , and we obtain  $P(G + H) = \max(\alpha(G), \alpha(H)) + 1$ .  $\square$

**Corollary 3.7.** *For any complete multi-partite graph  $G$ , we have  $p(G) = 2$ , and  $P(G)$  is one more than the size of the largest partite set of  $G$ .*

*Proof.* This follows immediately from Theorem 3.6 and the fact that  $G$  is the join of one partite set and either another partite set or a complete multi-partite subgraph.  $\square$

## 4 Products with Complete Graphs

The Cartesian product of two graphs  $G$  and  $H$ , denoted  $G \times H$ , has vertex set

$$V(G \times H) = \{(g, h) : g \in V(G), h \in V(H)\},$$

and there is an edge between vertices  $(g, h)$  and  $(g', h')$  if  $g = g'$  and  $h$  and  $h'$  are adjacent in  $H$  or if  $h = h'$  and  $g$  and  $g'$  are adjacent in  $G$ . We often view a Cartesian product as consisting of  $|H|$  copies of  $G$  with additional edges between different copies. Computing the pebbling numbers of a Cartesian product is one of the primary directions in the pebbling literature (see [7]). In this section, we study the pegging numbers of the Cartesian product  $G \times K_n$ . The main result of this section, Theorem 4.3, provides an upper bound for the pegging number based on pebbling numbers. (A formal definition of pebbling is given below.) Before presenting Theorem 4.3, we first study the pegging and optimal pegging numbers of the Cartesian product  $K_m \times K_n$ .

**Proposition 4.1.** *For positive integers  $m$  and  $n$ , not both 1, the pegging number of  $K_m \times K_n$  is*

$$P(K_m \times K_n) = \alpha(K_m \times K_n) + 1 = \min\{m, n\} + 1.$$

*Proof.* Without loss of generality, we may assume  $m \geq n$ . Since  $\alpha(K_m \times K_n) = n$ , we know  $P(K_m \times K_n) \geq n + 1$ . Let  $D$  be any distribution of  $n + 1$  pegs on  $K_m \times K_n$  and let  $t$  be any target vertex; we may assume  $t \notin D$ .

Let  $A$  be the copy of  $K_m$  containing  $t$ . If  $A$  has pegs on two vertices  $u$  and  $v$ , then  $u \xrightarrow{v} t$  puts a peg on  $t$ . If  $A$  only has one peg, say on  $u$ , then some other copy  $B$  of  $K_m$  has pegs on two vertices  $v$  and  $w$ . Let  $x$  be the vertex in  $A$  adjacent to  $w$ . Then the moves  $v \xrightarrow{w} x$  (which may be a stacking move) and (if  $x$  is not  $t$ )  $x \xrightarrow{u} t$  (which may be a pebbling move) put a peg on  $t$ .

Finally, if  $A$  has no pegs, either two copies of  $K_m$  each have (at least) two pegs or one copy  $B$  has pegs at three vertices  $u$ ,  $v$ , and  $w$ . In the first case, one move puts a peg on  $A$ , reducing to the case when  $A$  has one peg. In the second case, let  $x$  be the vertex in  $B$  adjacent to  $t$ . If  $x$  equals  $u$ ,  $v$ , or  $w$ , one move puts a peg on  $t$ . If not, then  $u \xrightarrow{v} x$  and  $w \xrightarrow{x} t$  put a peg on  $t$ . Thus  $t \in \text{Reach}(D)$  and  $P(K_m \times K_n) = n + 1$ .  $\square$

**Proposition 4.2.** *For positive integers  $m$  and  $n$ , the optimal pegging number of  $K_m \times K_n$  is*

$$p(K_m \times K_n) = \begin{cases} 1 & \text{if } mn = 1, \\ 2 & \text{if } mn > 1 \text{ and } \min\{m, n\} \leq 2, \\ 3 & \text{otherwise.} \end{cases}$$

*Proof.* Without loss of generality, we may assume  $m \geq n$ . The result is trivial for  $n = 1$ . When  $n = 2$ , place pegs on two vertices in one copy of  $K_n$ . Any vertex can be reaching in one move, showing that  $p(K_m \times K_n) = 2$ .

Finally, suppose  $n \geq 3$ . Any placement of two pegs on  $K_m \times K_n$  leaves a copy of  $K_m$  and a copy of  $K_n$  with no pegs, so we cannot peg to their common vertex. Thus  $p(K_m \times K_n) > 2$ . On the other hand, place pegs at three vertices  $u$ ,  $v$ , and  $w$  in one copy  $A$  of  $K_m$  and let  $t$  be any vertex. If  $t$  is in  $C$  or just adjacent to  $u$ ,  $v$ , or  $w$ , one move puts a peg at  $t$ . Otherwise, let  $x$  be the vertex in  $C$  adjacent to  $t$ . Then  $u \xrightarrow{v} x$  and  $w \xrightarrow{x} t$  put a peg on  $t$ . So  $p(K_m \times K_n) = 3$ .  $\square$

We now move on to the main result of this section. Consider a graph  $G$  with some number of pebbles placed on each vertex. A *pebbling move* consists of removing two pebbles from one vertex and placing one pebble on an adjacent vertex. The *pebbling number*  $f(G)$  (respectively, the *two-pebbling number*  $f_2(G)$ ) of a graph  $G$  is the smallest positive integer  $p$  so that, given any distribution of  $p$  pebbles on  $G$  and any vertex  $t$ , there exists a sequence of pebbling moves that places a pebble at  $t$  (respectively, two pebbles at  $t$ ). Note that  $f(G) \geq |G|$  and that  $f(G)$  and  $f_2(G)$  are taken to be infinite if  $G$  is disconnected.

**Theorem 4.3.** *For any graph  $G$  and positive integer  $n$ , the pegging number of  $G \times K_n$  satisfies  $P(G \times K_n) \leq f_2(G)$ . Furthermore,  $P(G \times K_2) \leq \max\{f(G), |G| + 1\}$ .*

*Proof.* The result is trivial when  $G$  is disconnected, so we may assume  $G$  is connected. If  $v$  is a vertex in  $G$ , let  $K_v$  denote the copy of  $K_n$  in  $G \times K_n$  consisting of  $n$  copies of  $v$ . If  $w$  is a vertex in  $K_n$ , let  $G_w$  denote the copy of  $G$  in  $G \times K_n$  consisting of  $|G|$  copies of  $w$ . Let  $\pi : V(G \times K_n) \rightarrow V(G)$  be the projection map.

Let  $D$  be any distribution of  $f_2(G)$  pegs on  $G \times K_n$  and let  $t$  be any target vertex. Let  $g = \pi(t)$  and view the multi-distribution  $\pi(D)$  as a distribution of pebbles on  $G$ . There are  $f_2(G)$  pebbles in  $\pi(D)$ , so there is a sequence of pebbling moves on  $G$  that places two

pebbles on  $g$ . For each pebbling move on  $G$ , perform a corresponding stacking or pebbling move on  $G \times K_n$  (that is, given a pebbling move from  $h \in G$  to  $k \in G$ , perform a stacking move from two vertices in  $K_h$  to a vertex in  $K_k$  or a pebbling move from a vertex in  $K_h$  to a vertex in  $K_k$ ). This puts two pegs on  $K_g$ . One more stacking or pebbling move puts a peg on  $t$ . By Theorem 2.4,  $t \in \text{Reach}(D)$ . Thus  $P(G \times K_n) \leq f_2(G)$ .

Now let  $p = \max\{f(G), |G| + 1\}$ , let  $D$  be any distribution of  $p$  pegs on  $G \times K_2$ , and let  $t$  be any target vertex (not in  $D$ ). Let  $g = \pi(t)$  and view the multi-distribution  $\pi(D)$  as a distribution of pebbles on  $G$ . Let  $u$  be the other vertex in  $K_g$  besides  $t$ .

If  $K_g$  does not contain any pegs, then we can move a pebble to  $g$ , since there are  $p \geq f(G)$  pebbles on  $G$ . Perform corresponding stacking or pebbling moves on  $G \times K_2$  to move a peg to  $K_g$ , always taking the target of each move to be in the copy of  $G$  containing  $t$ . (Since we only stack pegs in this copy, we will never have to make a pebbling move using two pegs in the other copy of  $G$ . The final move will place a peg at  $t$ .)

On the other hand, suppose  $K_g$  does have a peg (on  $u$ ). Since  $p > |G|$ , some vertex  $h \neq g$  has at least two pebbles. Choose the nearest such  $h$  to  $g$  and let  $g = x_0, x_1, x_2, \dots, x_k = h$  be a minimum length path  $P$  from  $g$  to  $h$ . Note that  $x_1, \dots, x_{k-1}$  each have at most one pebble. If they each have exactly one pebble, perform the sequence of pebbling moves  $(m_k, m_{k-1}, \dots, m_1)$ , where  $m_i$  removes two pebbles from  $x_i$  and puts one pebble on  $x_{i-1}$ . Any corresponding sequence of stacking and pebbling moves on  $G \times K_2$  puts another peg in  $K_g$ , and  $t$  can be reached with (at most) one more stacking or pebbling move.

If, however, not all of the vertices  $x_1, \dots, x_{k-1}$  have a pebble, let  $j \geq 1$  be the least index such that  $x_j$  does not have a pebble. Then we can move a pebble to  $x_j$ . Note that  $x_0, x_1, \dots, x_{j-1}$  each have only one pebble, so if any of these pebbling moves involved one of these vertices, the first such move would have to land a pebble on  $x_i$  for some  $i$  with  $0 \leq i < j$ . After this move,  $x_1, \dots, x_{i-1}$  have one pebble and  $x_i$  has two pebbles, and as in the previous paragraph, we can reach  $t$ . Hence we may assume that none of  $x_0, x_1, \dots, x_{j-1}$  are involved in moving a pebble to  $x_j$ . As before, using an appropriate corresponding sequence of stacking and pebbling moves, we can get a peg to the vertex  $u$  of  $K_{x_j}$  that is adjacent to the vertex  $v$  of  $K_{x_{j-1}}$  having a peg. Now we can make a pegging move from  $u$  over  $v$  to a vertex of  $K_{x_{j-2}}$  (or to  $t$  for  $j = 1$ ), and again we are reduced to the case of the previous paragraph, from which we can reach  $t$ .

In every case, we can reach  $t$  by making stacking and pebbling moves from the distribution  $D$ , so Theorem 2.4 shows that  $P(G \times K_2) \leq p$ , proving the theorem.  $\square$

## 5 Hypercubes

For each positive integer  $n$ , the hypercube  $Q_n$  is the graph  $K_2 \times K_2 \times \dots \times K_2$ , a product of  $n$  copies of  $K_2$ . For convenience, we label the vertices of each factor  $K_2$  with 0 and 1, and we label each of the  $2^n$  vertices of  $Q_n$  with the corresponding binary  $n$ -tuple. An early result of Fan Chung in pebbling literature is that the pebbling number of  $Q_n$  is  $2^n$  (see [2]). Using this fact and Theorem 4.3, we can compute the pegging numbers of hypercubes.

**Theorem 5.1.** *For any positive integer  $n$ , we have  $P(Q_n) = 2^{n-1} + 1$ .*

*Proof.* We have  $f(Q_{n-1}) = 2^{n-1} = |Q_{n-1}|$  (see [2]), so Theorem 4.3 gives  $P(Q_n) = P(Q_{n-1} \times K_2) \leq 2^{n-1} + 1$ . On the other hand, the set consisting of the  $2^{n-1}$  vertices of  $Q_n$  for which the sum of the  $n$  coordinates is even is an independent set, so we have  $P(Q_n) \geq \alpha(Q_n) + 1 \geq 2^{n-1} + 1$ . This proves the theorem.  $\square$

It seems harder to compute the exact optimal pegging number of a hypercube, but we can give upper and lower bounds that are correct up to a polynomial factor.

**Lemma 5.2.** *Given any distributions  $D$  and  $E$  on graphs  $G$  and  $H$ , respectively,*

$$\text{Reach}(D) \times \text{Reach}(E) \subseteq \text{Reach}(D \times E),$$

*where  $D \times E$  is viewed as a distribution on  $G \times H$ .*

*Proof.* Let  $t \in \text{Reach}(D)$  and  $u \in \text{Reach}(E)$ . We must show that  $(t, u) \in \text{Reach}(D \times E)$ . For each  $v \in E$ , make the moves on the subgraph  $G \times \{v\}$  using  $D \times \{v\}$  needed to put a peg on  $(t, v)$ . Now there is a peg on each vertex in  $\{t\} \times E$ , so we can make moves in  $\{t\} \times H$  to put a peg on  $(t, u)$ .  $\square$

**Corollary 5.3.** *For any graphs  $G$  and  $H$  we have  $p(G \times H) \leq p(G)p(H)$ .*

Now we obtain our bounds on the optimal pegging numbers of cubes. The results and methods are similar to those of Moews in the case of pebbling (see [9]). The upper bound was originally obtained by Lenhard L. Ng in 1994.

**Theorem 5.4.** *For any positive integer  $n$ ,*

$$\left(\sqrt{5} - 1\right)^n \leq p(Q_n) \leq (2n)^{3/2} \left(\sqrt{5} - 1\right)^n.$$

*Proof.* To prove the lower bound, let  $D$  be any distribution on  $Q_n$  with  $\text{Reach}(D) = V(Q_n)$ . By Lemma 2.1,  $\text{wt}_t(D) \geq 1$  for all  $t \in V(Q_n)$ . Summing over all the vertices of  $Q_n$  shows

$$\begin{aligned} 2^n &\leq \sum_{t \in Q_n} \text{wt}_t(D) \\ &= \sum_{t \in Q_n} \sum_{v \in D} \text{wt}_t(v) \\ &= \sum_{v \in D} \sum_{t \in Q_n} \text{wt}_t(v). \end{aligned}$$

The contribution of the peg at  $v \in D$  to this sum is

$$\begin{aligned} \sum_{t \in Q_n} \text{wt}_t(v) &= \sum_{i=0}^n \sum_{\substack{t \in Q_n \\ d(v,t)=i}} \text{wt}_t(v) \\ &= \sum_{i=0}^n \binom{n}{i} \sigma^i \\ &= (1 + \sigma)^n \\ &= \phi^n, \end{aligned}$$

with  $\phi = (1 + \sqrt{5})/2$ . Inserting this into the previous inequality shows

$$|D|\phi^n \geq 2^n$$

and

$$|D| \geq \left(\frac{2}{\phi}\right)^n = (\sqrt{5} - 1)^n,$$

which gives the lower bound.

In order to obtain the upper bound, we fix an integer  $n \geq 2$ , let  $r = \lceil n/(\phi + 2) \rceil$ , and let  $m = n - r$ . Then there is a binary linear code  $C$  in  $Q_m$  with covering radius at most  $r$  and dimension at most  $d = m(1 - H(r/m)) + (3/2) \lg m + 1$ , with  $H(x) = -x \lg x - (1-x) \lg(1-x)$  and  $H(0) = H(1) = 0$  (see [4]; “lg” here means logarithm to the base 2). That is, there is a (linear) subset  $C$  of  $Q_m$  with at most  $2^d$  vertices such that every vertex of  $Q_m$  has distance at most  $r$  from some vertex of  $C$ .

Now let  $D$  be the distribution  $Q_r \times C$  on  $Q_r \times Q_m = Q_n$  and take any vertex  $(u, v) \in Q_r \times Q_m$ . We have a vertex  $w \in C$  at distance at most  $r$  from  $v$  in  $Q_m$ . This means that some copy of  $Q_r$  in  $Q_m$  contains both  $v$  and  $w$ , so that the sub-distribution  $Q_r \times \{w\} \subseteq D$  and the vertex  $(u, v)$  both lie in some copy of  $Q_r \times Q_r$ . Now note that

$$\text{Reach}(Q_r \times Q_0) = Q_r \times Q_r;$$

this follows immediately by induction on  $r \geq 1$ , using Lemma 5.2. Thus  $(u, v)$  is in the reach of the sub-distribution  $Q_r \times \{w\}$ , which means that all of  $Q_n$  is in the reach of the distribution  $D$ .

This gives us  $p(Q_n) \leq |Q_r \times C| \leq 2^{r+d}$ , and we have

$$\begin{aligned} r + d &= r + m \left(1 - H\left(\frac{r}{m}\right)\right) + \frac{3}{2} \lg m + 1 \\ &= \frac{3}{2} \lg m + n + 1 - (n - r)H\left(\frac{r}{n - r}\right) \cdot \left(\frac{m - r}{m}\right)^{m - r}. \end{aligned}$$

Now  $H(x)$  is increasing on the interval  $[0 \dots 1/2]$ , so we have

$$\begin{aligned} r + d &\leq \frac{3}{2} \lg m + n + 1 - (n - r)H\left(\frac{n/(\phi + 2)}{n - n/(\phi + 2)}\right) \\ &= \frac{3}{2} \lg m + n + 1 - (n - r)H\left(\frac{1}{\phi + 1}\right) \\ &\leq \frac{3}{2} \lg m + n + 1 - \left(n - \frac{n}{\phi + 2} - 1\right) H(\phi^{-2}) \\ &\leq \frac{3}{2} \lg n + n + 1 + \left[\lg\left(1 - \frac{1}{\phi + 2}\right) + H(\phi^{-2})\right] - n \left(\frac{\phi + 1}{\phi + 2}\right) H(\phi^{-2}) \\ &\leq \frac{3}{2} \lg n + n + 1 + \frac{1}{2} - n \left(\frac{\phi^2}{\phi + 2}\right) (2\phi^{-2} \lg \phi + \phi^{-1} \lg \phi) \end{aligned}$$

$$\begin{aligned}
&= \frac{3}{2} \lg(2n) + n - n \lg \phi \\
&= \frac{3}{2} \lg(2n) + n \lg(\sqrt{5} - 1).
\end{aligned}$$

We conclude that

$$p(Q_n) \leq (2n)^{3/2} (\sqrt{5} - 1)^n,$$

as desired.  $\square$

## 6 Graphs of Small Diameter

We can prove sharp bounds on the pegging numbers and optimal pegging numbers of graphs of small diameter. Our results will require merely that the graphs have vertex-edge diameter (defined below) at most some value  $d$ , which is weaker than requiring the diameter to be at most  $d$ .

We define the distance between a vertex and an edge of a graph to be the minimum distance between the vertex and an endpoint of the edge. We denote the diameter of a graph  $G$  by  $d(G)$  and the *vertex-edge diameter* (the maximum distance between a vertex and an edge) of a non-null graph  $G$  by  $d_{\text{ve}}(G)$ . (Recall that a graph is non-null if it has at least one edge). Note that a non-null graph  $G$  of diameter  $d$  has  $d - 1 \leq d_{\text{ve}}(G) \leq d$ . Notice also that the vertex-edge diameter of a graph, unlike the diameter, can be increased by adding an edge; for instance adding an edge to  $K_{1,3}$  increases the vertex-edge diameter from 1 to 2.

Let  $G$  be a non-null graph with  $d_{\text{ve}}(G) \leq 1$ . From the definition of optimal pegging number, we have  $p(G) = 2$ , since if we put pegs on any two adjacent vertices, we can jump to any other vertex in one move. We claim that we also have  $P(G) = \alpha(G) + 1$ . For if we are given any target vertex  $t \in G$  and a distribution  $D$  of  $\alpha(G) + 1$  vertices not containing  $t$ , then  $D$  contains two adjacent vertices, one of which is adjacent to  $t$ , and we may jump the peg on one of these vertices over the other one to  $t$ .

Now let  $G$  be a graph of order at least 2 and radius 1, so that it has a vertex  $u$  adjacent to every other vertex. Then  $G$  has optimal pegging number 2, since  $u$  and any other vertex together have reach equal to  $G$ . We claim that we also have  $P(G) = \alpha(G) + 1$ . If  $G$  is complete, the result is trivial, so we may assume  $\alpha(G) \geq 2$ . Given any distribution  $D$  of at least  $\alpha(G) + 1$  vertices,  $D$  contains two adjacent vertices, so we can jump one over the other to reach  $u$  (if  $u$  is not already in  $D$ ). Since we have  $|D| \geq 3$ , we can still jump from another vertex over  $u$  to any desired target (not already in  $D$ ). This proves the claim.

We now obtain sharp bounds on the optimal pegging number and pegging number of graphs of vertex-edge diameter 2.

**Proposition 6.1.** *The optimal pegging number of any graph  $G$  with  $d_{\text{ve}}(G) = 2$  is at most 4.*

*Proof.* Let  $G$  be a graph with  $d_{\text{ve}}(G) = 2$ . Now  $G$  must have two disjoint edges  $(u, v)$  and  $(w, x)$ , and we let  $D$  be  $\{u, v, w, x\}$ . Take any target vertex  $t \in G$ . If  $t$  is in  $D$  or is adjacent



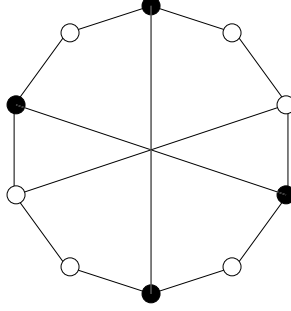


Figure 1: The graph above has vertex-edge diameter 2. The distribution consisting of the black vertices has reach equal to the entire graph, so the optimal pegging number is at most 4. It can be shown that no distribution of 3 vertices can reach every vertex, so the optimal pegging number is exactly 4.

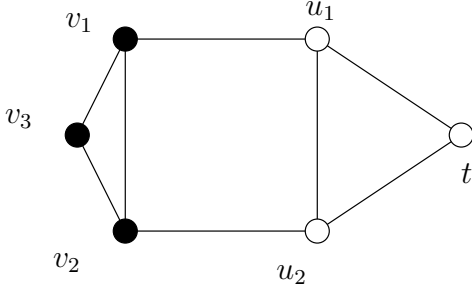
to a vertex of  $D$ , then we can reach  $t$  in at most one move. Otherwise, either  $u$  or  $v$ , say  $u$ , has a common neighbor  $y$  with  $t$ . We may assume that  $y$  is not adjacent to  $w$  or  $x$  and then, similarly,  $w$ , say, must have a common neighbor  $z$  with  $y$ . Now we can jump from  $v$  over  $u$  to  $y$ , from  $x$  over  $w$  to  $z$ , and from  $z$  over  $y$  to  $t$ , reaching the target. Thus  $G$  is in the reach of  $D$  and we have  $p(G) \leq 4$ .  $\square$

The authors experimented by computing the optimal pegging numbers of a number of small graphs of vertex-edge diameter 2. It seemed empirically that such graphs that do not contain a pair of adjacent vertices dominating the graph have optimal pegging number 3, however one example with optimal pegging number 4 is given in Figure 1. While there do not seem to be any simple examples of graphs with diameter 2 and optimal pegging number 4, there is the Hoffman–Singleton graph. This is the unique (up to isomorphism) 7-regular graph of order 50, diameter 2, and girth 5 (so any two non-adjacent vertices have exactly one common neighbor) (see [6]).

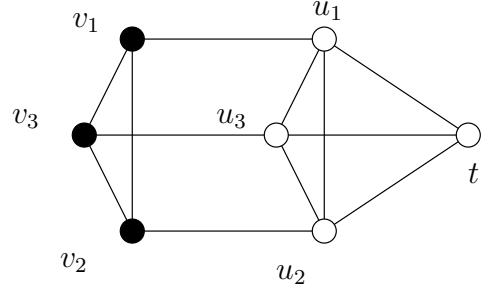
Suppose we have some distribution of three vertices  $u, v, w$  on the Hoffman–Singleton graph. There are two possibilities up to symmetry: either  $u$  and  $w$  are both adjacent to  $v$ , or  $u, v, x, w$ , and  $y$  are the five vertices of a pentagon listed in order, for some vertices  $x$  and  $y$ . In the first case, it is not possible to make two moves in sequence, so the total reach of the distribution consists of 20 vertices: the 3 vertices  $u, v, w$ , plus 6 neighbors each of  $u$  and  $w$  that are not  $v$ , plus 5 neighbors of  $v$  that are not  $w$  or  $u$ . In the second case, it is possible to make at most two moves in sequence, but only if the first move is  $u \xrightarrow{v} x$  or  $v \xrightarrow{u} y$ . Thus, the reach of the second distribution consists of the 5 vertices  $\{u, v, x, w, y\}$ , together with their 25 other neighbors, making the total number of vertices in the reach 30. Hence the optimal pegging number of the Hoffman–Singleton graph is 4.

For the pegging numbers of graphs of diameter 2, we obtain results similar to those of Clarke, Hochberg, and Hurlbert (see [3]).

**Theorem 6.2.** *For any integer  $\alpha \geq 2$ , the maximum pegging number of graphs of vertex-edge diameter 2 and independence number  $\alpha$  is  $\alpha + 2$ , and this bound can be achieved by a*



(a)



(b)

Figure 2: Let  $G$  be a graph with  $\alpha(G) = \alpha \geq 2$  and with pegging number  $\alpha + 2$ . If  $G$  has vertex-edge diameter 2, then  $G$  contains (a) as an induced subgraph, and if  $G$  has diameter 2, then  $G$  contains (b) as an induced subgraph. A vertex is black or white according to whether it lies in the distribution in the proof of Theorem 6.2.

*graph of diameter 2. Furthermore, given a distribution of at least this size for such a graph, we always can reach any target with at most 3 moves. Finally, any such graph with pegging number exactly  $\alpha + 2$  contains the graph of Figure 2(a) (respectively, Figure 2(b)) as an induced subgraph.*

*Proof.* First note that the graph in Figure 3 has both diameter and vertex-edge diameter equal to 2 and has independence number  $\alpha$ , but has pegging number at least  $\alpha + 2$ .

Now suppose we have a graph  $G$  with  $d_{ve}(G) = 2$ , a target vertex  $t$  of  $G$ , and a distribution  $D$  of at least  $\alpha(G) + 2$  pegs on  $G$ . Then there are two adjacent vertices  $u$  and  $v$  of  $D$ , and one of them, say  $u$ , has distance at most 2 from  $t$ . We may assume the distance is 2, so  $t$  and  $u$  have a common neighbor  $w$ . Now move a peg from  $v$  over  $u$  to  $w$ . The remaining distribution still has two adjacent pegs (and we may assume that neither is  $w$ ), so we may, as before, make a second move to get a peg adjacent to  $w$ . Finally, with a third move, we jump this peg over  $w$  to  $t$ . Thus, given a graph  $G$  of vertex-edge diameter 2 and any distribution of at least  $\alpha(G) + 2$  pegs, any target can be reached with at most 3 moves, and we have  $P(G) \leq \alpha(G) + 2$ .

Finally, suppose we have a graph  $G$  with  $d_{ve}(G) = 2$  and  $P(G) = \alpha(G) + 2$ . Then we have a target vertex  $t$  of  $G$  and a distribution  $D$  of  $\alpha(G) + 1$  pegs whose reach does not contain  $t$ . Notice that no vertex of  $D$  can be equal to or adjacent to  $t$ , or we would be able to get a peg to  $t$  in at most two moves, as in the previous paragraph. Similarly, there cannot exist two disjoint pairs of adjacent vertices of  $D$ , or we would be able to get to  $t$  in at most three moves. Let  $v_1$  and  $v_2$  be two adjacent vertices of  $D$ ; then  $(D \setminus \{v_1\}) \cup \{t\}$  has more than  $\alpha(G)$  vertices and so contains two adjacent vertices, which must lie in  $D \setminus \{v_1\}$ . One of these two vertices must be  $v_2$ , lest  $D$  contain two disjoint adjacent pairs, so we have  $v_2$  adjacent to some vertex  $v_3 \in D \setminus \{v_1\}$ . Similarly,  $(D \setminus \{v_2\}) \cup \{t\}$  contains an adjacent pair

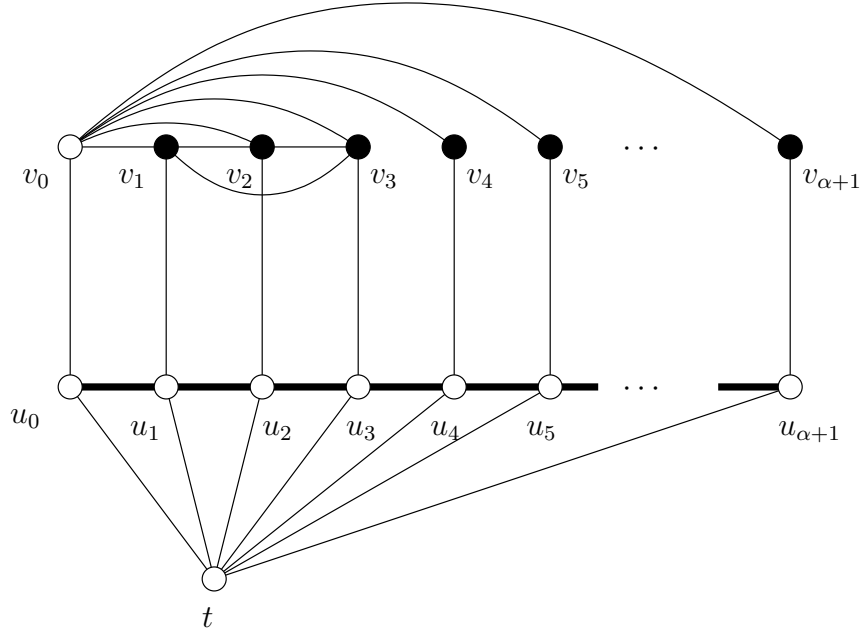


Figure 3: The graph above both has diameter and vertex-edge diameter equal to 2 and has independence number  $\alpha$ . The vertices  $u_i$  form a complete graph on  $\alpha + 2$  vertices (the thick lines denote the many edges joining these vertices), and the  $\alpha + 1$  vertices that are colored black represent a distribution of pegs from which it is not possible to reach  $t$ . In fact, the only pegging moves that leave adjacent pegs involve moving a peg to  $v_0$ , and then no more useful moves can be made. Thus the pegging number of this graph is at least  $\alpha + 2$ .

of vertices, and this pair must consist of  $v_1$  and  $v_3$ . Thus  $v_1, v_2, v_3$  are all adjacent, and these are the only adjacencies in  $D$ .

Now one of  $v_1$  and  $v_2$ , say  $v_1$ , must have a common neighbor  $u_1$  with  $t$ , and  $u_1$  cannot be adjacent to any vertex  $v \in D \setminus \{v_1\}$ , for then we would be able to jump from  $v_2$  (or  $v_3$  if  $v$  is  $v_2$ ) over  $v_1$  to  $u_1$  and then from  $v$  over  $u_1$  to  $t$ . Similarly, one of  $v_2$  and  $v_3$ , say  $v_2$ , has a common neighbor  $u_2$  with  $t$ , and  $u_2$  cannot be adjacent to any vertex of  $D \setminus \{v_2\}$ . Finally, the set  $(D \setminus \{v_1, v_2\}) \cup \{u_1, u_2\}$  contains two adjacent vertices, and these must be  $u_1$  and  $u_2$ , so  $u_1$  and  $u_2$  are adjacent. Thus the subgraph of  $G$  induced by  $\{t, u_1, u_2, v_1, v_2, v_3\}$  is isomorphic to the graph of Figure 2(a). In addition, if  $G$  has diameter 2, then there must, similarly, be a vertex  $u_3$  adjacent to  $t, v_3, u_1, u_2$  and not adjacent to  $v_1, v_2$ , so  $G$  contains an induced subgraph isomorphic to the graph of Figure 2(b).  $\square$

We now consider graphs with pegging number at most 3. First of all,  $K_0$  and  $K_1$  are the only graphs with pegging numbers 0 and 1, respectively. Also, since the pegging number of a non-null graph is greater than its independence number, we see that  $2K_1$  and  $K_n$  for  $n \geq 2$  are the only graphs with pegging number 2. We can now use Theorem 6.2 to classify graphs with pegging number 3. Recall that we sometimes identify  $D$  with the subgraph of  $G$  spanned by  $D$ .

**Corollary 6.3.** *A graph  $G$  has pegging number 3 if and only if  $G$  is isomorphic to  $3K_1$ , or  $G$  has independence number 2 and the following condition fails:*

- (\*)  *$G$  is spanned by  $C \cup D$  with  $C$  and  $D$  complete subgraphs of  $G$ , and either  $G$  is isomorphic to  $2K_1$  or  $2K_2$ , or  $G$  has vertices  $t \in C$  and  $v_1, v_2, v_3 \in D$ , such that  $t$  is not adjacent to any vertex of  $D$ , and every vertex of  $C$  is adjacent to at most one of  $v_1, v_2, v_3$ .*

*Proof.* We know that  $3K_1$  has pegging number 3, while other graphs with independence number at least 3 have pegging number at least 4. Also,  $2K_1$  and  $2K_2$  have pegging numbers 2 and 4, respectively, while all other graphs satisfying the forbidden condition (\*) have pegging number at least 4, since  $t$  is not in the reach of the distribution  $\{v_1, v_2, v_3\}$ . Thus, in order to prove the corollary, we may assume that we have  $\alpha(G) = 2$  and  $P(G) > 3$ , and we must show that the forbidden condition (\*) holds.

We first claim that  $G$  is spanned by a disjoint union of complete subgraphs. If  $G$  has diameter at least 3, then take two vertices  $x$  and  $y$  at distance at least 3 from each other. Then every vertex of  $G \setminus \{x, y\}$  is adjacent to exactly one of  $x$  and  $y$ , and  $x$  together with its neighbors form a complete subgraph of  $G$ , as do  $y$  and its neighbors. Thus we may assume that  $G$  has diameter at most 2. Now by (the proof of) Theorem 6.2, we know that  $G$  has vertices  $t, v_1, v_2, v_3$  such that  $t$  is not in the reach of the distribution  $\{v_1, v_2, v_3\}$ . Thus no neighbor of  $t$  is adjacent to more than one of  $v_1, v_2, v_3$ , and any two neighbors of  $t$  have a common non-neighbor among  $v_1, v_2, v_3$  and are, therefore, adjacent. Then  $t$  and its neighbors form a complete subgraph of  $G$ , as do the non-neighbors of  $t$ . This proves the claim.

Now  $G$  is spanned by a disjoint union of complete subgraphs  $C$  and  $D$ , and there are vertices  $t, v_1, v_2, v_3 \in G$  with  $t$  not in the reach of  $\{v_1, v_2, v_3\}$ . We may assume that  $t$  is

in  $C$ , and then at least two of  $v_1, v_2, v_3$ , say  $v_1$  and  $v_2$ , are in  $D$ , since otherwise we would be able to reach  $t$  in one move. First suppose that  $v_3$  is in  $D$ . If any vertex  $x \in G$  were either in  $C$  and adjacent to, say,  $v_1$  and  $v_2$ , or in  $D$  and adjacent to  $t$ , then we could move  $v_3 \xrightarrow{v_1} x, v_2 \xrightarrow{x} t$ . Thus neither of these occurs, and  $(*)$  holds. Now suppose that  $v_3$  is in  $C$ . If any vertex  $x \in G$  were either in  $C$  and adjacent to, say,  $v_1$ , or in  $D$  and adjacent to  $v_3$ , then we could move  $v_2 \xrightarrow{v_1} x, x \xrightarrow{v_3} t$ . Therefore, neither  $v_1$  nor  $v_2$  is adjacent to any vertex of  $C$ , and  $v_3$  is not adjacent to any vertex of  $D$ . Now if  $D$  has a vertex  $y$  besides  $v_1$  and  $v_2$ , then  $(*)$  holds with  $v_3$  replaced by  $y$  and  $t$  replaced by  $v_3$ . Thus we may assume that  $D$  has order 2, so there are no edges between the vertices of  $C$  and  $D$ . Then either  $G$  is isomorphic to  $2K_2$ , or  $C$  has order at least 3, and in the latter case, interchanging the roles of  $C$  and  $D$  shows that  $(*)$  holds. This proves the corollary.  $\square$

Finally, we obtain a sharp upper bound on the pegging numbers of graphs of vertex-edge diameter 3.

**Theorem 6.4.** *For any integer  $\alpha \geq 2$ , the maximum pegging number of graphs of vertex-edge diameter 3 and independence number  $\alpha$  is  $2\alpha + 1$ , and this bound can be achieved by a graph of diameter 3. Furthermore, given a distribution of at least this size for such a graph, we can always reach any target in at most 7 moves.*

First we need three lemmas that will be used several times in the proof of this theorem. The proofs of these results also introduce several ideas that will appear in the proof of the theorem.

**Lemma 6.5.** *Let  $G$  be a connected graph with  $|G| > 3$  and  $|G| \geq 2\alpha(G) - 1$ . Then  $G$  contains a subgraph isomorphic to the path  $P_4$ .*

*Proof.* Let  $v$  be a vertex of  $G$  of degree at least 2, with neighbors  $u$  and  $w$ . If every neighbor of  $v$  had degree 1, then  $G$  would be isomorphic to the complete bipartite graph  $K_{1,|G|-1}$ . But then we would have

$$|G| \geq 2\alpha(G) - 1 = 2(|G| - 1) - 1 = 2|G| - 3,$$

which gives  $|G| \leq 3$ , contrary to hypothesis. Hence some neighbor of  $v$ , say  $w$ , has degree at least 2 and is adjacent to a vertex besides  $v$ . If  $w$  is adjacent to  $u$ , then, as  $G$  is connected and contains some vertex besides  $u, v, w$ , one of these three vertices, say  $w$ , is adjacent to a fourth vertex. Thus, in any case, we may assume that  $w$  is adjacent to some vertex  $x$  not equal to  $u$  or  $v$ . Then  $G$  contains the path  $uvw x$  of order 4, proving the lemma.  $\square$

**Lemma 6.6.** *Let  $G$  be a graph with  $d_{ve}(G) = 3$ , let  $t \in G$  be a target vertex, and let  $D \subset G$  be a distribution not containing  $t$ , such that  $D$  contains a subgraph  $P$  isomorphic to  $P_4$ . Then we can reach a neighbor of  $t$  in at most 3 moves using only the pegs of  $P$ .*

*Proof.* Label the vertices of  $P$  in order as  $u, v, w, x$ . Now either  $v$  or  $w$ , say  $v$ , has distance at most 3 from  $t$ . If this distance is 1, we are done, and if it is 2, we can jump from  $u$  over  $v$  to a neighbor of  $t$ , so we may assume that  $v$  has distance 3 from  $t$ . Then  $v$  has a

neighbor  $y$  adjacent to a neighbor  $z$  of  $t$ , and we can make the sequence of pegging moves  $u \xrightarrow{v} y$ ,  $x \xrightarrow{w} v$ ,  $v \xrightarrow{y} z$ , reaching a neighbor of  $t$  in at most 3 moves using only the pegs of  $P$ , as desired.  $\square$

**Lemma 6.7.** *Let  $G$  be a graph with  $d_{\text{ve}}(G) = 3$ , let  $t \in G$  be a target vertex, let  $D \subset G$  be a distribution not containing  $t$ , and let  $S \subseteq G \setminus D$  be an independent set satisfying the following conditions:*

- *No vertex of  $S$  is adjacent to a vertex of  $D$ .*
- *No neighbor of a vertex of  $S$  both has distance 2 from  $t$  and is adjacent to a vertex of a component of  $D$  isomorphic to  $K_3$ .*
- *We have  $|D| + d + 2|S| \geq 2\alpha(G) + 1$ , with  $d$  being the number of isolated vertices of the subgraph  $D$  of  $G$ .*

*Then we can reach a neighbor of  $t$  by making at most 3 moves using the pegs of  $D$ .*

*Proof.* We may assume that every vertex of  $D$  has distance at least 2 from  $t$  and that every non-isolated vertex of  $D$  has distance at least 3 from  $t$ , since otherwise we can reach a neighbor of  $t$  in at most one move. For each component  $C$  of  $D$  that is isomorphic to  $K_3$ , take a vertex  $v_C$  of  $C$  at distance 3 from  $t$  and a neighbor  $u_C$  (in  $G \setminus D$ ) of  $v_C$  at distance 2 from  $t$ . For any component  $C$  of  $D$ , we let  $\hat{C}$  denote either the subgraph of  $G$  induced by  $C \cup \{u_C\}$ , for  $C \cong K_3$ , or  $C$ , otherwise. We will refer to  $\hat{C}$  for  $C \cong K_3$  as an “extended subgraph”.

If some such vertex  $u_C$  were adjacent to a vertex  $w$  of  $D$  besides  $v_C$ , then we could jump from some vertex of  $C$  over  $v_C$  to  $u_C$  and then from  $w$  over  $u_C$  to a neighbor of  $t$ , so we may assume that this is not the case. Similarly, if  $u_C$  were adjacent to a vertex  $u_{C'}$  corresponding to another component  $C'$ , then we could jump over  $v_C$  to  $u_C$ , over  $v_{C'}$  to  $u_{C'}$ , and from  $u_{C'}$  over  $u_C$  to a neighbor of  $t$ , so we may assume that this does not occur, either. Thus the extended subgraphs have independence number 2, and the components of  $E = D \cup \{u_C \mid C \cong K_3 \text{ a component of } D\}$  consist of the graphs  $\hat{C}$ , for  $C$  a component of  $D$ . Notice also that, by the hypotheses, no vertex of  $S$  is adjacent to a vertex of  $E$ .

We now have

$$\begin{aligned}
|D| &> 2\alpha(G) - d - 2|S| \\
&\geq 2\alpha(E \cup S) - d - 2|S| \\
&= 2\alpha(E) + 2|S| - d - 2|S| \\
&= 2\alpha(E) - d
\end{aligned}$$

and, therefore,

$$\sum_{C \text{ a component of } D} (|C| - 2\alpha(\hat{C})) = |D| - 2\alpha(E) > -d$$



and

$$\sum_{\substack{C \text{ a component of } D \\ |C| > 1}} (|C| - 2\alpha(\hat{C})) > 0.$$

Hence we have a component  $K$  of  $D$  with  $|K| > 1$  and  $|K| > 2\alpha(\hat{K})$ . Now  $K$  cannot be isomorphic to  $K_3$ , since then  $\hat{K}$  would be one of the extended subgraphs above and would have independence number 2. Thus  $K = \hat{K}$  has order at least 4. Now by Lemma 6.5,  $K$  contains a subgraph isomorphic to  $P_4$ , and by Lemma 6.6, we can get a peg to a neighbor of  $t$  in at most 3 moves using only the pegs of  $K \subseteq D$ . This proves the lemma.  $\square$

*Proof of Theorem 6.4.* First, we exhibit a family of graphs achieving the maximum pegging number. Given a value  $\alpha \geq 2$ , we define a graph  $G$  as follows: The vertices of  $G$  are  $\{v_{ij} \mid 2 \leq i \leq \alpha \text{ and } 1 \leq j \leq 2, \text{ or } (i, j) \in \{(1, 1), (\alpha, 3), (\alpha, 4)\}\} \cup \{u_{ij} \mid 1 \leq i, j \leq \alpha, i \neq j\}$ . All vertices with first coordinate  $i$  are adjacent, and, in addition,  $u_{ij}$  and  $u_{ji}$  are adjacent. Let  $t$  be  $v_{11}$  and  $D$  be the set of all other  $v_{ij}$ . Then  $G$  is a graph of diameter 3 and independence number  $\alpha$ , and  $D$  is a set of  $2\alpha$  vertices whose reach does not include  $t$ . Thus we have  $P(G) \geq 2\alpha(G) + 1$ .

Now suppose that we have a graph  $G$  with  $\alpha(G) = \alpha$  and  $d_{\text{ve}}(G) = 3$ , together with a target vertex  $t$  and a distribution  $D$  of at least  $2\alpha + 1$  vertices. We wish to show that we can reach  $t$  from  $D$  in at most 7 moves. We may assume that  $D$  does not contain  $t$ . Suppose that  $D$  contains a neighbor  $t'$  of  $t$  or a non-isolated (in  $D$ ) vertex adjacent to a neighbor  $t'$  of  $t$ . By making one move if necessary, we may reduce to the first case with  $|D| \geq 2\alpha$ . Now if  $t'$  had a neighbor in  $D$ , we could reach  $t$  in one (more) move, so assume not. Take  $S = \{t'\}$  and  $D' = D \setminus S$ . Then we have

$$|D'| + 2|S| = |D| + 1 \geq 2\alpha + 1,$$

so by Lemma 6.7, we know that we can make at most 3 pegging moves not involving  $t'$  to get a peg to a neighbor of  $t'$ . Then we can jump this peg over  $t'$  to  $t$  to reach  $t$  in a total of at most 5 moves. Thus we may assume that every vertex of  $D$  has distance at least 2 from  $t$  and that every non-isolated vertex of  $D$  has distance at least 3 from  $t$ .

As in the proof of Lemma 6.7, for each component  $C$  of  $D$  that is isomorphic to  $K_3$ , we take a vertex  $v_C \in C$  at distance 3 from  $t$  and a neighbor  $u_C$  of  $v_C$  at distance 2 from  $t$ . As before, we denote by  $\hat{C}$  the extended subgraph  $C \cup \{u_C\}$  for such components  $C$  (and  $C$  itself for  $C$  not isomorphic to  $K_3$ ). Suppose that some vertex  $u_C$  is adjacent to a vertex  $w \in D \setminus C$ . Then we can jump from a vertex of  $C$  over  $v_C$  to  $u_C$  and from  $w$  over  $u_C$  to a neighbor  $t'$  of  $t$ . Now let  $D'$  be the current distribution minus  $t'$ , with  $d'$  isolated vertices, and let  $S$  be  $\{t'\}$ . If  $t'$  has a neighbor in  $D'$ , then we can get to  $t$  in one more move, so we may assume it does not. We have  $d' \geq 1$ , since the remaining vertex of  $C$  is isolated in  $D'$ , and we obtain

$$|D'| + d' + 2|S| \geq |D| - 3 + 1 + 2 \geq 2\alpha + 1.$$

Now we can apply Lemma 6.7 to  $t', D', S$  to show that we can reach a neighbor of  $t'$  with at most 3 moves of the pegs of  $D'$ , giving a total of at most 5 moves. Finally, we can jump

from this neighbor of  $t'$  over  $t'$  to  $t$ , reaching  $t$  in at most 6 moves. Thus we may assume that no vertex  $u_C$  is adjacent to any vertex of  $D \setminus C$ .

Now suppose that some vertex  $u_C$  is adjacent to a vertex  $w$  of  $C$  besides  $v_C$ . As before, we can jump from the other vertex of  $C$  over  $v_C$  to  $u_C$  and then from  $w$  over  $u_C$  to a neighbor  $t'$  of  $t$ . Let  $D'$  be the current distribution minus  $t'$  and  $S$  be  $\{t', w\}$ . Then  $w$  is not adjacent to any vertex of  $D'$ , and we may assume that  $t'$  is not either, or we would be able to get to  $t$  in one more move. Now if some neighbor  $x$  of  $w$  were adjacent to a vertex of some  $C'$ , then we could peg to  $x$ , from  $x$  over  $w$  to  $u_C$ , and from  $u_C$  over  $t'$  to  $t$ , so we may assume that this is not the case. Thus no neighbor of  $w$  violates the second condition of Lemma 6.7, and we have

$$|D'| + 2|S| \geq |D| - 3 + 4 \geq 2\alpha + 2,$$

so we can apply Lemma 6.7 as before to show that we can reach  $t$  in a total of at most 6 moves. Hence we may assume that no vertex  $u_C$  is adjacent to any vertex of  $D$  besides  $v_C$ .

Finally, suppose two vertices  $u_C$  and  $u_{C'}$  with  $C \neq C'$  are adjacent. Then we can jump over  $v_C$  to  $u_C$ , over  $v_{C'}$  to  $u_{C'}$ , and from  $u_{C'}$  over  $u_C$  to a neighbor  $t'$  of  $t$ . Let  $D'$  be the current distribution minus  $t'$ , with  $d'$  isolated vertices, and let  $S$  be  $\{t'\}$ . As before, we have  $d' \geq 2$ , we may assume that  $t'$  has no neighbor in  $D'$ , and we calculate

$$|D'| + d' + 2|S| \geq |D| - 4 + 2 + 2 \geq 2\alpha + 1.$$

By applying Lemma 6.7, we can reach  $t$  in at most 4 more moves, giving a total of at most 7 moves. Thus we may assume that no two vertices  $u_C$  and  $u_{C'}$  are adjacent.

Hence, as in the proof of Lemma 6.7, the extended subgraphs all have independence number 2, and the components of the graph  $E = D \cup \{u_C \mid C \cong K_3 \text{ a component of } D\}$  consist of the graphs  $\hat{C}$  for  $C$  a component of  $D$ . Again, as in the proof of Lemma 6.7, we have

$$\begin{aligned} |D| &\geq 2\alpha(G) + 1 \\ &\geq 2\alpha(E) + 2\alpha(\{t\}) + 1 \\ &= 2\alpha(E) + 3 \end{aligned}$$

and, therefore,

$$\sum_{C \text{ a component of } D} (|C| - 2\alpha(\hat{C})) \geq 3. \quad (1)$$

Thus we have a component  $K$  of  $D$  with  $|K| > 2\alpha(\hat{K})$ . Again,  $K$  cannot be isomorphic to  $K_3$ , as  $\hat{K}$  would then be an extended subgraph with independence number 2. Therefore  $K$  must equal  $\hat{K}$  and must have order at least 4. By Lemma 6.5,  $K$  then contains a subgraph isomorphic to  $P_4$ .

For any (not necessarily induced) subgraph  $P$  of  $D$  isomorphic to  $P_4$ , let  $E'$  be obtained from  $D \setminus P$  by extending each component  $C$  isomorphic to  $K_3$  with a vertex  $u_C$  as before. For each component  $C$  of  $D \setminus P$ , we let  $\check{C}$  be  $C \cup \{u_C\}$  if  $C$  is isomorphic to  $K_3$  and  $C$  otherwise, paralleling the notation  $\hat{C}$  for components of  $D$ . Suppose that some extended

subgraph  $\check{C}$  does not have independence number 2 or is not a component of  $E'$  or that some component  $C$  of  $D \setminus P$  satisfies  $|C| > 3$  and  $|C| \geq 2\alpha(C) - 1$ . Then either some vertex  $u_C$  is adjacent to a vertex of  $D \setminus P$  besides the corresponding  $v_C$  or is adjacent to a vertex  $u_{C'}$  for  $C' \neq C$ , or, by Lemma 6.5,  $C$  must contain a subgraph isomorphic to  $P_4$ . As above, or by applying Lemma 6.6, we can, in any case, use the pegs of  $D \setminus P$  to get to a neighbor  $t'$  of  $t$  in at most 3 moves. Then, by Lemma 6.6, we can use the pegs of  $P$  to get to a neighbor of  $t'$  in at most 3 more moves, after which we can jump over  $t'$  to  $t$ , thereby reaching  $t$  in a total of at most 7 moves. Thus we may assume that all extended subgraphs coming from  $D \setminus P$  have independence number 2, that the components of  $E'$  consist of the subgraphs  $\check{C}$  for  $C$  a component of  $D \setminus P$ , and that no component  $C$  of  $D \setminus P$  has  $|C| > 3$  and  $|C| \geq 2\alpha(C) - 1$ .

Therefore, the only components  $C$  of  $D \setminus P$  with  $|C| - 2\alpha(\check{C}) \geq -1$  have order at most 3 and, therefore, are isomorphic to  $K_1$ ,  $K_2$ ,  $P_3$ , or  $K_3$ . Notice that  $|C| - 2\alpha(\check{C})$  is 0 for  $C \cong K_2$  and is  $-1$  for the other possibilities for  $C$  of order at most 3; in particular, this difference is never positive. Now, as  $D \setminus P$  has at least  $2\alpha - 3$  vertices, a calculation analogous to that producing Equation (1) gives

$$\sum_{C \text{ a component of } D \setminus P} (|C| - 2\alpha(\check{C})) \geq -1.$$

Thus, for any subgraph  $P$  of  $D$  isomorphic to  $P_4$ , all components of  $D \setminus P$  are isomorphic to  $K_2$ , except for at most one component isomorphic to  $K_1$ ,  $P_3$ , or  $K_3$ . In particular, this applies to all components of  $D$  besides  $K$ . Therefore, Equation (1) yields

$$|K| \geq 2\alpha(K) + 3. \quad (2)$$

We now have three cases, based on the order of  $K$ .

*Case 1:* We have  $|K| \leq 6$ . Then  $\alpha(K)$  is 1,  $|K|$  is 5 or 6, and  $K$  is complete. Label five of the vertices of  $K$  as  $v_1, \dots, v_5$ . As before, some vertex of  $K$ , say  $v_1$ , has distance 3 from  $t$ , and we can take a path  $v_1 u w t$  of length 3 from  $v_1$  to  $t$ . Suppose that  $u$  is not adjacent to any vertex of  $E \setminus K$ . Then  $K \cup \{u\}$  is still a component of  $E \cup \{u\}$ , and  $u$  is not adjacent to  $t$ , so we can apply the argument leading to Equations (1) and (2), with  $E$  replaced by  $E \cup \{u\}$  and  $\hat{K} = K \cup \{u\}$ , to obtain  $|K| \geq 2\alpha(K \cup \{u\}) + 3$ . This gives  $\alpha(K \cup \{u\}) = 1$ , so that  $K \cup \{u\}$  is complete. Thus we may reach  $t$  in 4 moves with the sequence  $v_2 \xrightarrow{v_1} u, v_3 \xrightarrow{u} w, v_5 \xrightarrow{v_4} u, u \xrightarrow{w} t$ . Therefore, we may assume that  $u$  is adjacent to a vertex  $x$  of  $E \setminus K$ . Then we can move  $v_2 \xrightarrow{v_1} u$  and  $x \xrightarrow{u} w$  (the latter preceded by another move to get a peg to  $x$  if we had  $x = u_C$ ), and then we get to  $t$  via the sequence  $v_4 \xrightarrow{v_3} v_1, v_5 \xrightarrow{v_1} u, u \xrightarrow{w} t$ , reaching  $t$  in at most 6 moves, as desired.

*Case 2:* We have  $|K| = 7$ . Then  $\alpha(K)$  is at most 2. First suppose that  $K$  is spanned by a disjoint union of  $P_2$  and  $P_5$ , with the endpoints of the  $P_5$  non-adjacent. Label the vertices of the  $P_2$  as  $v$  and  $v'$  and the vertices of the  $P_5$  as  $v_1, \dots, v_5$ , with  $v_i$  and  $v_{i+1}$  adjacent, and  $v_1$  and  $v_5$  non-adjacent. Now either  $v$  or  $v'$ , say  $v$ , has distance 3 from  $t$ , and we can take a path  $v u w t$  of length 3 from  $v$  to  $t$ . If  $u$  is not adjacent to any vertex of  $E \setminus K$ , then  $K \cup \{u\}$  is a component of  $E \cup \{u\}$ , so, as in Case 1, we get  $|K| \geq 2\alpha(K \cup \{u\}) + 3$  and  $\alpha(K \cup \{u\}) = 2$ ,

and  $u$  must be adjacent to either  $v_1$  or  $v_5$ , say  $v_5$ . Therefore, in any event,  $u$  must be adjacent to some vertex  $x$  of  $E \setminus \{v, v', v_1, \dots, v_4\}$ . We can now move  $v' \xrightarrow{v} u$  and  $x \xrightarrow{u} w$  (the latter preceded by another move if we had  $x = u_C$ ) and then use  $v_1, \dots, v_4$  to reach  $t$  in 4 more moves, for a total of at most 7 moves. Thus we may assume that  $K$  is not spanned by a disjoint union of  $P_2$  and  $P_5$ , with the endpoints of the  $P_5$  non-adjacent.

Let  $P$  be a path in  $K$  of maximal length. Suppose that  $P$  omitted at least one vertex of  $K$ . If the endpoints of  $P$  were not adjacent, then any vertex  $v \in K \setminus P$  would be adjacent to one of them (since we have  $\alpha(K) \leq 2$ ), and we could extend  $P$ . If the endpoints of  $P$  were adjacent, then the vertices of  $P$  would form a cycle, some vertex  $v \in K \setminus P$  would be adjacent to a vertex of  $P$  (since  $K$  is connected), and, again, we could extend  $P$ . In either case, we contradict the maximality of the length of  $P$ , so  $P$  cannot omit any point of  $P$  (and is, therefore, a Hamiltonian path of  $K$ ).

Label the vertices of  $P$  (and, therefore, of  $K$ ) as  $w_1, \dots, w_7$ , with  $w_i$  and  $w_{i+1}$  adjacent. Since  $K$  is spanned by the union of the two paths  $w_1w_2$  and  $w_3 \dots w_7$ , we must have  $w_3$  and  $w_7$  adjacent, and, similarly,  $w_1$  and  $w_5$  must be adjacent. Next, the paths  $w_3w_4$  and  $w_2w_1w_5w_6w_7$  show that  $w_2$  and  $w_7$  are adjacent, as are  $w_1$  and  $w_6$ . Now the paths  $w_2w_7$  and  $w_3w_4w_5w_1w_6$  show that  $w_3$  and  $w_6$  are adjacent, as are  $w_2$  and  $w_5$ . Finally, the paths  $w_3w_4$  and  $w_1w_2w_5w_6w_7$  show that  $w_1$  and  $w_7$  are adjacent. Now one of  $w_1$  and  $w_7$ , say  $w_1$ , has distance 3 from  $t$ , and we can take a path  $w_1uwt$  of length 3 from  $w_1$  to  $t$ . Finally, we can reach  $t$  in 6 moves via the sequence  $w_7 \xrightarrow{w_1} u$ ,  $w_3 \xrightarrow{w_2} w_1$ ,  $w_1 \xrightarrow{u} w$ ,  $w_4 \xrightarrow{w_5} w_1$ ,  $w_6 \xrightarrow{w_1} u$ ,  $u \xrightarrow{w} t$ , as desired.

*Case 3:* We have  $|K| \geq 8$ . Let  $P = v_1, \dots, v_4$  be a path on 4 vertices in  $K$ . We know that all components of  $K \setminus P$  are isomorphic to  $K_2$ , except for at most one component isomorphic to  $K_1$ ,  $P_3$ , or  $K_3$ . Suppose that one of  $v_1$  and  $v_2$  is adjacent to a vertex of a component of  $K \setminus P$  of order at least 2, and that the same holds for one of  $v_3$  and  $v_4$ . There are at least two components of  $K \setminus P$  of order at least 2 (since  $K$  has order at least 8), and each one is adjacent to a vertex of  $P$  (since  $K$  is connected), so we can pick distinct such components  $C$  and  $C'$  with  $v_1$  or  $v_2$  adjacent to a vertex of  $C$  and  $v_3$  or  $v_4$  adjacent to a vertex of  $C'$ . Then the subgraph of  $K$  induced by  $\{v_1, v_2\} \cup C$  contains a subgraph  $P'$  isomorphic to  $P_4$ , and  $K \setminus P'$  has a component of order at least 4 (namely, the one containing  $\{v_3, v_4\} \cup C'$ ), contradicting the assumption about the complement of any  $P_4$  in  $K$ . Thus, by symmetry, we may assume that neither  $v_3$  nor  $v_4$  is adjacent to a vertex of a component of  $K \setminus P$  of order at least 2.

Suppose that no component of  $D \setminus P$  is isomorphic to  $K_3$ . Then we have

$$\begin{aligned}
2\alpha(D) - 1 &= 2\alpha(D) + 2\alpha(\{t\}) - 3 \\
&\leq 2\alpha(G) - 3 \\
&\leq |D| - 4 \\
&= |D \setminus P| \\
&= \sum_{C \text{ a component of } D \setminus P} |C|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{C \text{ a component of } D \setminus P} 2\alpha(C) \\
&= 2\alpha(D \setminus P),
\end{aligned}$$

giving  $\alpha(D) = \alpha(D \setminus P)$ . This implies that every vertex of  $P$  is adjacent to a vertex of  $D \setminus P$ . Applying this to  $v_3$  and  $v_4$  shows that  $D \setminus P$  must have a component isomorphic to  $K_1$ , consisting of a single vertex  $x$ , and that both  $v_3$  and  $v_4$  must be adjacent to this vertex  $x$ . Now the vertices  $v_1$  and  $v_2$ , together with the vertices of some component of  $K \setminus P$  isomorphic to  $K_2$  (which must have a vertex adjacent to either  $v_1$  or  $v_2$ , since  $K$  is connected), form a subgraph of  $K$  isomorphic to  $P_4$ , whose complement in  $K$  contains the complete subgraph induced by  $\{v_3, v_4, x\}$ . Thus, replacing  $P$  by this new path if necessary, we may assume that  $D \setminus P$  has a component isomorphic to  $K_3$ . In particular, no component of  $D \setminus P$  is isomorphic to either  $K_1$  or  $P_3$ , and neither  $v_3$  nor  $v_4$  is adjacent to any vertex of  $K \setminus P$ .

Let  $\{x, y\}$  be any component of  $K \setminus P$  isomorphic to  $K_2$ . If  $v_1$  were adjacent to, say,  $x$ , then the complement of the path  $xv_1v_2v_3$  in  $K$  would contain the two isolated vertices  $y$  and  $v_4$ , contradicting the assumption about the complement of any  $P_4$  in  $K$ . Therefore,  $v_1$  is not adjacent to any vertex of a component of  $K \setminus P$  isomorphic to  $K_2$ . Thus,  $x$ , say, is adjacent to  $v_2$ . The vertex  $v_1$  cannot be adjacent to any vertex of a component of  $K \setminus P$ , which would have to be the component isomorphic to  $K_3$ , since otherwise, the complement of the path  $yxv_2v_3$  in  $K$  would contain a component of order at least 4 (that containing  $v_1$  and the  $K_3$ ). Finally,  $v_2$  must be adjacent to a vertex of some other component  $C$  of  $K \setminus P$  besides  $\{x, y\}$ , and the complement in  $K$  of a path on  $x, v_2$ , and two vertices of  $C$  contains the singleton component  $\{y\}$  and the component containing  $v_1$ , which equals either  $\{v_1\}$  or  $\{v_1, v_3, v_4\}$ . This final contradiction of the condition on complements of  $P_4$  in  $K$  shows that, given our previous assumptions, Case 3 cannot occur, and this proves the theorem.  $\square$

## 7 Conclusion

We have introduced two new pegging quantities, namely the pegging number and optimal pegging number of a graph. We have successfully computed these numbers for many classes of graphs, including paths, cycles, joins, and products with complete graphs, using diverse tools including basic pegging lemmas and pebbling. In a forthcoming paper, Wood [11] studies pegging numbers of graph powers and products, develops new general lower bounds for the pegging number, studies the size of the reach of a distribution, and classifies some pegging moves as unnecessary.

Any progress in the computation of these numbers and in the development of computation tools would be of interest. In particular, what is the connection between the pegging numbers and other graph invariants such as girth and connectivity? Also, do pegging numbers behave nicely under other graph operations like graph composition? Are there any more connections between pegging and pebbling? It should be noted that the pegging analogue to Graham's conjecture [2] is false, as shown in [11].

In Theorem 2.4, we considered the effect of allowing stacking and pebbling moves in

pegging. Let us refer to these two types of moves as *pegging* moves. Let the pegging number (respectively, the optimal pegging number) of a graph be the smallest positive integer  $d$  such that every (respectively, some) multi-distribution  $D$  of size  $d$  has  $\text{Reach}_a(D) = V(G)$ .

Because pegging is like pebbling with more moves allowed, the pegging and optimal pegging numbers of a graph are at most its pebbling and optimal pebbling numbers, respectively. On the other hand, Theorem 2.4 shows that allowing stacking and pebbling moves does not help reach additional targets in pegging. Thus, because in pegging the starting configuration can be any multi-distribution, the pegging number of a graph is at least its pebbling number, and the optimal pegging number of a graph is at most its optimal pebbling number. All of these inequalities may be strict; for example the graph  $P_4$  has pebbling number 8, pegging number 3, and optimal pegging number 5; and the graph obtained by adding a pendant edge to each leaf of  $K_{1,3}$  has optimal pebbling number 4, optimal pegging number 4, and optimal pegging number 3. Many of the basic properties of pebbling, pegging, optimal pebbling, and optimal pegging numbers carry over to pegging and optimal pegging numbers, and it would be interesting to study these new quantities in more detail.

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